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Observer-Based Feedback-Control for the Stabilization of a Class of Parabolic Systems

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Abstract

We consider the stabilization of a class of linear evolution systems $z' = Az + Bv$ under the observation $y = Cz$ by means of a finite dimensional control v. The control is based on the design of a Luenberger observer which can be infinite or finite dimensional (of dimension large enough). In the infinite dimensional case, the operator A is supposed to generate an analytical semigroup with compact resolvent and the operators B and C are unbounded operators whereas in the finite dimensional case, A is assumed to be a self-adjoint operator with compact resolvent, B and C are supposed to be bounded operators. In both cases, we show that if (A, B) and (A, C) verify the Fattorini-Hautus Criterion, then we can construct an observer-based control v of finite dimension (greater or equal than largest geometric multiplicity of the unstable eigenvalues of A) such that the evolution problem is exponentially stable. As an application, we study the stabilization of the diffusion system.

Keywords Parabolic systems · Feedback control · Stabilization · Luenberger observers.

Mathematics Subject Classification 93B53 · 93D15 · 93C20

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1 Introduction and Main Results

Given $b \in L^2(0,1)$, consider the one-dimensional controlled heat equation

$$\begin{aligned} &\partial_t z(t,x) = \partial_{xx} z(t,x) + b(x)v(t), \quad t > 0, \quad x \in (0,1), \\ &\partial_x z(t,0) = 0, \quad z(t,1) = 0, \\ &z(0) = z^0. \end{aligned} \tag{1.1}$$

Obviously, the open-loop system (i.e. for $v = 0$) is exponentially stable, with a decay rate defined by the smallest eigenvalue of the underlying operator describing the free dynamics (namely the positive definite self-adjoint operator $-\partial_{xx}$ with Neumann boundary condition at $x = 0$ and Dirichlet boundary condition at $x = 1$). Based on the observation

$$y(t) = \int_0^1 c(x)z(t,x)\,\mathrm{d}x, \tag{1.2}$$

where $c \in L^2(0,1)$, a natural question that arises is to know whether it is possible to design a finite dimensional feedback control v, such that the closed-loop system (1.1) is exponentially stable with an arbitrary prescribed decay rate $\sigma > 0$. In a recent work, a positive and constructive answer to this question has been proposed by Katz and Fridman [13], using an observer-based feedback control. More precisely, the authors proposed feasible design conditions for the construction of such controls for a more general $1D$ reaction-diffusion equation with variable coefficients (i.e. for a free dynamics described by an operator of the form $\partial_x\left(p(x)\partial_x\cdot\right) - q(x)\cdot$).

In this paper, our objective is to generalize this result to a large class of parabolic systems, possibly multi-dimensional and involving unbounded control and/or observation operators. More precisely, given three complex Hilbert spaces $\mathbb{H}$ (the state space), $\mathbb{U}$ (the control space) and $\mathbb{Y}$ (the observation space), consider the linear infinite dimensional system

$$\begin{aligned} &z'(t) = Az(t) + Bv(t), \\ &z(0) = z^0, \\ &y(t) = Cz(t), \end{aligned} \tag{1.3}$$

where $A : \mathcal{D}(A) \longrightarrow \mathbb{H}$ is an unbounded operator, $B \in \mathcal{L}\left(\mathbb{U}, (\mathcal{D}(A^*))'\right)$ and $C \in \mathcal{L}\left(\mathcal{D}(A), \mathbb{Y}\right)$. Given $\sigma > 0$, the goal of this paper is to prove the existence of an observer-based control v such that the solution of (1.3) is exponentially stable, with a decay rate $-\sigma$:

$$\|z(t)\|_{\mathbb{H}} \leqslant Me^{-\sigma t}\|z^0\|_{\mathbb{H}}.$$

We will investigate two classes of infinite dimensional systems of the form (1.3). For the first one, the observer-based feedback is constructed using an infinite-dimensional observer (IDO). For the second one, under strongest assumptions on A, B and C, we design a finite dimensional observer (FDO) to construct the feedback control.

Of course, a finite dimensional observer can be considered as a particular case of infinite dimensional observer, but the former is much more interesting for computational purposes (and, clearly, more challenging to design). For instance, if the system (1.3) describes an evolution PDE (like the Reaction–Diffusion equation of Sect. 4), constructing an infinite dimensional observer requires to solve an evolution PDE, corresponding to the observer (1.7). On the contrary, the finite dimensional observer described by (1.10) is simply a set of evolution ODE. As one might expect, the proofs to get these two results use different techniques.

We will consider the two following sets of assumptions on A, B and C (below, $\rho(A)$ denotes the resolvent set of A):

- **Infinite-Dimensional Observer (IDO)**

$$\text{A is an operator with compact resolvent generating an analytic semigroup on } \mathbb{H}, \tag{H1.A}$$

$$(\mu_0 \operatorname{Id} - A)^{-\gamma} B \in \mathcal{L}(\mathbb{U}, \mathbb{H}) \quad \text{is a linear bounded operator for some } \gamma \in [0, 1) \text{ and } \mu_0 \in \rho(A), \tag{H1.B}$$

$$C(\mu_0 \operatorname{Id} - A)^{-\widehat{\gamma}} \in \mathcal{L}(\mathbb{H}, \mathbb{Y}) \quad \text{is a linear bounded operator for some } \widehat{\gamma} \in [0, 1) \text{ and } \mu_0 \in \rho(A), \tag{H1.C}$$

$$\begin{gathered} \forall \varepsilon \in \mathcal{D}(A^*),\ \forall \lambda \in \mathbb{C},\ \operatorname{Re} \lambda \geqslant -\sigma, \quad A^* \varepsilon = \overline{\lambda} \varepsilon \quad \text{and} \quad B^* \varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0, \\ \forall \varepsilon \in \mathcal{D}(A),\ \forall \lambda \in \mathbb{C},\ \operatorname{Re} \lambda \geqslant -\sigma, \quad A \varepsilon = \lambda \varepsilon \quad \text{and} \quad C \varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0. \end{gathered} \tag{H1.D}$$

- **Finite-Dimensional Observer (FDO)**

$$\text{A is a self-adjoint operator with compact resolvent,} \tag{H2.A}$$

$$B \in \mathcal{L}(\mathbb{U}, \mathbb{H}), \tag{H2.B}$$

$$C \in \mathcal{L}(\mathbb{H}, \mathbb{Y}), \tag{H2.C}$$

$$\begin{gathered} \forall \varepsilon \in \mathcal{D}(A),\ \forall \lambda \in \mathbb{R},\ \lambda \geqslant -\sigma, \quad A \varepsilon = \lambda \varepsilon \quad \text{and} \quad B^* \varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0, \\ \forall \varepsilon \in \mathcal{D}(A),\ \forall \lambda \in \mathbb{R},\ \lambda \geqslant -\sigma, \quad A \varepsilon = \lambda \varepsilon \quad \text{and} \quad C \varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0. \end{gathered} \tag{H2.D}$$

In this work, we provide a generalization of the result obtained in [13] for a class of parabolic systems. First, in Sect. 2, we consider the case of an infinite dimensional observer (IDO). More precisely, we prove Theorem 1.1: we split the system (1.3) into two parts in the same spirit as [2, 3], one part corresponds to the unstable modes of A that defines a finite dimensional system and the other part is the infinite dimensional system corresponding to the stable modes of A. Using (H1.D), we construct a Luenberger observer in the spirit of [16]. We prove the stability of the closed loop system since the state, the observer and the error define all together a triangular system. In Sect. 3, we deal with the case of the finite dimensional observer (FDO) and we prove Theorem 1.2: here, A is supposed to be a selfadjoint operator with compact resolvent. Hence, the projection operator defined in (1.5) becomes orthogonal. The first step will be also to split the system and to construct a finite dimensional observer using (H2.D) in the spirit of [13]. The challenge here consists to prove the stability of the observer, the error and the state that define a strongly coupled system this time. To deal with this issue, we show that it involves a Volterra type equation that can be solved by using a fixed point argument, the mapping defines a contraction if the observer dimension is large enough.

It is worth mentioning that assumption (H1.D) (and its counterpart (H2.D) in the self-adjoint case) is the well-known Fattorini-Hautus criterion for exponential stabilization (see [3, 8, 10]).

For every $\nu > 0$, we set

$$\Sigma_\nu^+ := \{\lambda_j \in \sigma(A) \mid \operatorname{Re}\lambda_j \geqslant -\nu\}, \quad \Sigma_\nu^- := \{\lambda_j \in \sigma(A) \mid \operatorname{Re}\lambda_j < -\nu\}, \tag{1.4}$$

where $\sigma(A)$ is the spectrum of A. Condition (H1.A) in the IDO case and (H2.A) in the FDO case imply that Σ_ν^+ describes a finite set. We define the projection

$$P_\nu^+ = -\frac{1}{2\iota\pi}\int_{\Gamma_\nu^+} (\zeta \operatorname{Id} - A)^{-1}\, d\zeta, \qquad P_\nu^- = \operatorname{Id} - P_\nu^+, \tag{1.5}$$

where Γ_ν^+ is a curve enclosing Σ_ν^+ but no other point of the spectrum of A and oriented counterclockwise (see [12, V.5, p.272]). We set

$$z_\nu^\pm := P_\nu^\pm z, \qquad \forall z \in \mathbb{H}.$$

We also introduce the finite dimensional operators

$$A_\nu^\pm := AP_\nu^\pm, \qquad B_\nu^\pm := P_\nu^\pm B, \qquad C_\nu^\pm := C\iota_\nu^\pm,$$

where $\iota_\nu^\pm$ is the embedding operator from $\mathbb{H}_\nu^\pm := P_\nu^\pm \mathbb{H}$ to $\mathbb{H}$. Finally, we denote by Q_ν^+ the orthogonal projection from $\mathbb{Y}$ onto $\mathbb{Y}_\nu^+ := CP_\nu^+\mathbb{H}$.

We are now in position to state our main results in the (IDO) and (FDO) cases.

1.1 Infinite-Dimensional Observer (IDO)

Theorem 1.1 *Let $\sigma > 0$ be given and assume that assumptions (H1.A), (H1.B), (H1.C) and (H1.D) hold true. Then, there exist two operators $K_\sigma^+ \in \mathcal{L}(\mathbb{H}, B^*(P_\sigma^+)^*\mathbb{H})$ and $L_\sigma^+ \in \mathcal{L}(CP_\sigma^+\mathbb{H}, P_\sigma^+\mathbb{H})$ such that the observer-based feedback control defined by*

$$v(t) = K_\sigma^+ \widehat{z}(t), \tag{1.6}$$

where the infinite dimensional observer $\widehat{z}$ solves

$$\begin{aligned} &\widehat{z}'(t) = A\widehat{z}(t) + Bv(t) + L_\sigma^+ Q_\sigma^+ (C\widehat{z}(t) - y(t)), \\ &\widehat{z}(0) = 0, \end{aligned} \tag{1.7}$$

ensures that for any $z^0 \in \mathbb{H}$, the solution z of the closed loop (1.3–1.6–1.7), that is

$$\begin{aligned} &z'(t) = Az(t) + BK_\sigma^+ \widehat{z}(t), \\ &z(0) = z^0, \end{aligned}$$

satisfies

$$\|z(t)\|_{\mathbb{H}} \leqslant Me^{-\sigma t}\|z^0\|_{\mathbb{H}}, \qquad \forall t > 0. \tag{1.8}$$

To prove this result, we introduce the error $e := z - \widehat{z}$ and we check that systems (1.3) and (1.7) yield

$$\begin{pmatrix} z \\ e \end{pmatrix}' = \begin{pmatrix} A + BK_\sigma^+ & -BK_\sigma^+ \\ 0 & A + L_\sigma^+ Q_\sigma^+ C \end{pmatrix} \begin{pmatrix} z \\ e \end{pmatrix}.$$

The result follows then by choosing the operators K_σ^+ and L_σ^+ in such a way that both $A + BK_\sigma^+$ and $A + L_\sigma^+ Q_\sigma^+ C$ generate analytic semigroups with decay rate less than $-\sigma$. This is achieved by solving two Riccati equations, using the method proposed by Badra-Takahashi [2] (for K_σ^+) combined to a duality argument (for L_σ^+).

1.2 Finite-Dimensional Observer (FDO)

In this case, the stabilizing control is based on a finite dimensional observer of size $\sigma^\star > 0$, where $\sigma^\star > \sigma$ and need to be chosen large enough (see 3.19).

Theorem 1.2 *Let $\sigma > 0$ be given and assume that assumptions (H2.A), (H2.B), (H2.C) and (H2.D) hold true. Let $K_\sigma^+ \in \mathcal{L}(\mathbb{H}, B^*P_\sigma^+\mathbb{H})$ and $L_\sigma^+ \in \mathcal{L}(CP_\sigma^+\mathbb{H}, P_\sigma^+\mathbb{H})$ be the operators defined in Theorem 1.1. Then, there exists $\sigma^\star > \sigma$ such that the observer-based feedback control defined by*

$$v(t) = K_\sigma^+ \widehat{z}_\star(t) \in B^*P_\sigma^+\mathbb{H} \subset B^*P_{\sigma^\star}^+\mathbb{H}, \tag{1.9}$$

where the finite dimensional observer $\widehat{z}_\star \in P^+_{\sigma^\star}\mathbb{H}$ *solves*

$$\begin{aligned} &\widehat{z}'_\star(t) = A^+_{\sigma^\star}\widehat{z}_\star(t) + B^+_{\sigma^\star}v(t) + L^+_\sigma Q^+_\sigma(C^+_{\sigma^\star}\widehat{z}_\star(t) - y(t)), \\ &\widehat{z}_\star(0) = 0, \end{aligned} \tag{1.10}$$

ensures that for any $z^0 \in \mathbb{H}$*, the solution* z *of the closed loop (1.3–1.9–1.10), that is*

$$\begin{aligned} &z'(t) = Az(t) + BK^+_\sigma\widehat{z}_\star(t), \\ &z(0) = z^0, \end{aligned}$$

satisfies

$$\|z(t)\|_{\mathbb{H}} \leqslant Me^{-\sigma t}\|z^0\|_{\mathbb{H}}, \qquad \forall t > 0. \tag{1.11}$$

To prove this result, we proceed as follows. Introducing the auxiliary variables

$$e := z^+_{\sigma^\star} - \widehat{z}_\star, \qquad \mathbb{X} = \begin{pmatrix} \widehat{z}_\star \\ e \end{pmatrix},$$

we show that the equations satisfied by the state z and the observer $\widehat{z}_\star$ yield

$$\begin{aligned} &\mathbb{X}'(t) = \mathbb{A}\mathbb{X}(t) + \mathbb{L}(z^-_{\sigma^\star}(t)), \\ &(z^-_{\sigma^\star})'(t) = A^-_{\sigma^\star}z^-_{\sigma^\star}(t) + B^-_{\sigma^\star}\mathbb{K}^+_\sigma\mathbb{X}(t), \end{aligned} \tag{1.12}$$

where

$$\mathbb{A} = \begin{pmatrix} A^+_{\sigma^\star} + B^+_{\sigma^\star}K^+_\sigma & -L^+_\sigma Q^+_\sigma C^+_{\sigma^\star} \\ 0 & A^+_{\sigma^\star} + L^+_\sigma Q^+_\sigma C^+_{\sigma^\star} \end{pmatrix}, \quad \mathbb{L}(z^-_{\sigma^\star}) = \begin{pmatrix} -L^+_\sigma Q^+_\sigma C z^-_{\sigma^\star} \\ L^+_\sigma Q^+_\sigma C z^-_{\sigma^\star} \end{pmatrix}, \quad \mathbb{K}^+_\sigma = (K^+_\sigma, 0).$$

We prove then that the matrix $\exp(t\mathbb{A})$ is exponentially stable with a decay rate less than $-\sigma$. Next, thanks to the first equation in (1.12), we use Duhamel's formula to express $\mathbb{X}$ in terms of $z^-_{\sigma^\star}$. Plugging the obtained relation in the second equation of (1.12), we obtain an integral equation for $z^-_{\sigma^\star}$. We use a fixed point argument to prove the well-posedness of this integral equation in the weighted space

$$L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star}) := \{f \in L^\infty(0,\infty;\mathbb{H}^-_{\sigma^\star}) \text{ such that } e^{\sigma(\cdot)}f(\cdot) \in L^\infty(0,\infty;\mathbb{H}^-_{\sigma^\star})\}.$$

This provides the expected result, that is the exponential decay of the controlled system with a decay rate less than $-\sigma$.

1.3 Related Works

The output feedback stabilization of linear and non linear time-invariant systems has been extensively studied in the last decades. For finite dimensional systems, we refer the interested reader to the survey [21] where necessary and sufficient conditions are given. In the infinite dimensional case, sufficient conditions can be found for instance

[7, Chapter 5] using Luenberger observers for the design of compensators. In particular, for bounded observation and control operators, such sufficient conditions are the exponential stabilizability and exponential detectability of the system (A, B, C). A complete list of references and new necessary and sufficient conditions can be found in the recent work [25]. As already mentioned, the closest reference to our work is [13], in which the authors considered the case of a one dimensional heat equation. Their strategy is based on a (modal) splitting of the system into two parts: a finite dimensional unstable one and a stable infinite dimensional one. A Luenberger observer of large enough dimension is then constructed and the stability of the closed loop system is proved using a Lyapunov function. Contrarily to the proof proposed here, the arguments used in [13] are valid only in dimension one and heavily rely on the type of the considered equation. Let us also mention that in [14], the authors used a similar approach to prove the stabilization of a one dimension convection diffusion equation in the case of a boundary control. The use of modal splitting for the stabilization of infinite dimensional systems has also been achieved in some specific settings, like Burgers equations [5, 22, 23], Navier–Stokes system [2, 3, 9, 18, 20], semi-linear wave equation [6] and population dynamics [16, 17].

1.4 Outline

In Sect. 2, we prove Theorem 1.1, which provides the stabilizing observer-based feedback-control through an infinite dimensional observer. In Sect. 3, we construct a finite dimensional observer to design a similar feedback-control. For this case, we need to assume that the operator A is self-adjoint and the control and observation operators are bounded. Finally, in Sect. 4 and in Sect. 5, these abstract results are applied to obtain a stabilizing control for Reaction–Diffusion systems. The paper ends with a brief concluding section.

2 Infinite Dimensional Observer

2.1 Spectral Decomposition of the System

In this section, we suppose that assumptions (H1.A), (H1.B), (H1.C) and (H1.D) hold true. We consider below a classical modal decomposition (it has been used, for instance, in [2, 3, 9, 20]) that we recall it for the sake of completeness. Let $\sigma > 0$. We first separate the spectrum of A into "unstable" and "stable" modes using the projection P_σ^+ defined in (1.5). We set

$$\mathbb{H}_\sigma^+ = P_\sigma^+ \mathbb{H}, \quad \mathbb{H}_\sigma^- = (\mathrm{Id} - P_\sigma^+)\mathbb{H}, \quad \mathbb{H} = \mathbb{H}_\sigma^+ \oplus \mathbb{H}_\sigma^-.$$

According to this projection, we set

$$A_\sigma^+ := A_{|\mathbb{H}_\sigma^+} : \mathbb{H}_\sigma^+ \to \mathbb{H}_\sigma^+, \quad A_\sigma^- := A_{|\mathbb{H}_\sigma^-} : \mathcal{D}(A) \cap \mathbb{H}_\sigma^- \to \mathbb{H}_\sigma^-.$$

Then the spectrum of A_σ^+ is exactly Σ_σ^+ and the spectrum of A_σ^- is exactly Σ_σ^- where Σ_σ^+ and Σ_σ^- are defined in (1.4). We denote by A^* the adjoint operator of A and we define similarly the projection $(P_\sigma^+)^*$ such that

$$(P_\sigma^+)^* = -\frac{1}{2\imath\pi}\int_{\Gamma_\sigma^+}(\zeta \operatorname{Id} - A^*)^{-1}\, d\zeta. \tag{2.1}$$

The projection (2.1) provides also the following spaces

$$(\mathbb{H}_\sigma^+)^* = (P_\sigma^+)^*\mathbb{H}, \quad (\mathbb{H}_\sigma^-)^* = (\operatorname{Id} - (P_\sigma^+)^*)\mathbb{H}, \quad \mathbb{H} = (\mathbb{H}_\sigma^+)^* \oplus (\mathbb{H}_\sigma^-)^*, \tag{2.2}$$

with

$$(A_\sigma^+)^* := A^*_{|(\mathbb{H}_\sigma^+)^*} : (\mathbb{H}_\sigma^+)^* \to (\mathbb{H}_\sigma^+)^*, \quad (A_\sigma^-)^* := A_{|(\mathbb{H}_\sigma^-)^*} : \mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^* \to (\mathbb{H}_\sigma^-)^*.$$

Lemma 2.1 *There exist $\varepsilon > 0$ and $M > 0$ such that for any $\delta \geqslant 0$, $t > 0$*

$$\begin{aligned}
&\left\| e^{A_\sigma^- t}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant Me^{-(\sigma+\varepsilon)t}, && \left\|(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta e^{A_\sigma^- t}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant \frac{M}{t^\delta} e^{-(\sigma+\varepsilon)t},\\
&\left\| e^{(A_\sigma^-)^* t}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant Me^{-(\sigma+\varepsilon)t}, && \left\|(\mu_0 \operatorname{Id} - (A_\sigma^-)^*)^\delta e^{(A_\sigma^-)^* t}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant \frac{M}{t^\delta} e^{-(\sigma+\varepsilon)t}.
\end{aligned} \tag{2.3}$$

Proof We detail the proof only for the operator A_σ^-, as the arguments for its adjoint are similar. The first inequality is obvious. Concerning the second one, we first note that

$$(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta e^{A_\sigma^- t} = (\mu_0 \operatorname{Id} - A_\sigma^-)^\delta \left(A_\sigma^-\right)^{-\delta} \left(A_\sigma^-\right)^{\delta} e^{A_\sigma^- t}.$$

Applying [15, Corollary 6.11] with $B = (\mu_0 \operatorname{Id} - A_\sigma^-)^\delta$, $A = A_\sigma^-$ and $x = \left(A_\sigma^-\right)^{-\delta} y$, for $y \in \mathbb{H}_\sigma^-$, we obtain that for some constant positive C,

$$\left\|(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta \left(A_\sigma^-\right)^{-\delta} y\right\|_{\mathbb{H}} \leqslant C\|y\|_{\mathbb{H}}, \qquad \forall y \in \mathbb{H}_\sigma^-,$$

and thus

$$\left\|(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta \left(A_\sigma^-\right)^{-\delta}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant C.$$

Consequently,

$$\left\|(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta e^{A_\sigma^- t}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant \left\|(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta \left(A_\sigma^-\right)^{-\delta}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \left\|\left(A_\sigma^-\right)^{\delta} e^{A_\sigma^- t}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)},$$

and the desired estimate follows then immediately from [15, Theorem 6.13]. □

We also define

$$\mathbb{U}_\sigma^+ := B^*(\mathbb{H}_\sigma^+)^*, \quad \mathbb{U}_\sigma^- := B^*\left(\mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^*\right),$$

and

$$p_\sigma^+ : \mathbb{U} \to \mathbb{U}_\sigma^+, \quad p_\sigma^- : \mathbb{U} \to \mathbb{U}_\sigma^-, \quad i_\sigma^+ : \mathbb{U}_\sigma^+ \to \mathbb{U}, \quad i_\sigma^- : \mathbb{U}_\sigma^- \to \mathbb{U},$$

the orthogonal projections and the inclusion maps. Note that we have the following relations for the above maps:

$$i_\sigma^+ = (p_\sigma^+)^*, \quad i_\sigma^- = (p_\sigma^-)^*. \tag{2.4}$$

From [20], we can extend P_σ^+ and $(\mathrm{Id} - P_\sigma^+)$ as bounded operators

$$P_\sigma^+ \in \mathcal{L}(\mathcal{D}(A^*)', \mathbb{H}_\sigma^+), \quad (\mathrm{Id} - P_\sigma^+) \in \mathcal{L}(\mathcal{D}(A^*)', \left[\mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^*\right]').$$

We can thus define

$$B_\sigma^+ := P_\sigma^+ B i_\sigma^+ \in \mathcal{L}(\mathbb{U}_\sigma^+, \mathbb{H}_\sigma^+), \quad B_\sigma^- := (\mathrm{Id} - P_\sigma^+) B i_\sigma^- \in \mathcal{L}(\mathbb{U}_\sigma^-, \left[\mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^*\right]').$$

We show as in [2, 3, 20] that

$$P_\sigma^+ B = B_\sigma^+ p_\sigma^+, \quad (\mathrm{Id} - P_\sigma^+) B = B_\sigma^- p_\sigma^-.$$

Using the projections P_σ^+ and $\mathrm{Id} - P_\sigma^+$, system (1.3) can be split into the two subsystems (see [2, 3, 20]):

$$(z_\sigma^+)'(t) = A_\sigma^+ z_\sigma^+(t) + B_\sigma^+ p_\sigma^+ v(t), \quad z_\sigma^+(0) = P_\sigma^+ z^0, \tag{2.5}$$
$$(z_\sigma^-)'(t) = A_\sigma^- z_\sigma^-(t) + B_\sigma^- p_\sigma^- v(t), \quad z_\sigma^-(0) = (\mathrm{Id} - P_\sigma^+) z^0. \tag{2.6}$$

We also introduce the orthogonal projections Q_σ^+ from $\mathbb{Y}$ into $\mathbb{Y}_\sigma^+ = C\mathbb{H}_\sigma^+$ and we define

$$C_\sigma^+ = C\iota_\sigma^+,$$

where ι_σ^+ designates the injection operator from $\mathbb{H}_\sigma^+$ to $\mathbb{H}$. We are now in position to prove Theorem 1.1.

2.2 Construction of an Infinite Dimensional Observer-Based Control

Let us consider first the system

$$\begin{aligned} \widetilde{z}'(t) &= A\widetilde{z}(t) + Bu(t), \\ \widetilde{z}(0) &= \widetilde{z}^0. \end{aligned} \tag{2.7}$$

We want to construct a finite dimensional vector u such that the system (2.7) is exponentially stable. Let $N_\sigma \in \mathbb{N}^*$ and $(w_j)_{1\leqslant j\leqslant N_\sigma} \subset \mathbb{U}$, and let us suppose that the control $u(t)$ is of the form

$$u(t) = \sum_{j=1}^{N_\sigma} u_j(t) w_j,$$

where $u_j(t) \in \mathbb{C}$, for $1 \leqslant j \leqslant N_\sigma$ and $t \geqslant 0$. It is natural to introduce the mapping

$$\mathfrak{B} : \mathbb{C}^{N_\sigma} \longrightarrow \mathcal{D}(A^*)', \quad \Theta = (\theta_1, \cdots, \theta_{N_\sigma}) \longmapsto \mathfrak{B}\Theta = \sum_{j=1}^{N_\sigma} \theta_j B w_j,$$

in such a way that setting

$$\mathfrak{u}(t) := (u_1(t), \cdots, u_{N_\sigma}(t)),$$

system (2.7) is equivalent to

$$\begin{aligned} &\widetilde{z}'(t) = A\widetilde{z}(t) + \mathfrak{B}\mathfrak{u}(t), \\ &\widetilde{z}(0) = \widetilde{z}^0. \end{aligned} \tag{2.8}$$

It is worth noticing that the adjoint $\mathfrak{B}^* \in \mathcal{L}(\mathcal{D}(A^*), \mathbb{C}^{N_\sigma})$ is given by

$$\mathfrak{B}^*\psi = \Big(\langle w_1, B^*\psi\rangle_{\mathbb{U}}, \cdots, \langle w_{N_\sigma}, B^*\psi\rangle_{\mathbb{U}} \Big).$$

Using the projection P_σ^+, we get that (2.8) is equivalent to

$$\begin{aligned} &(\widetilde{z}_\sigma^+)'(t) = A_\sigma^+ \widetilde{z}_\sigma^+(t) + \mathfrak{B}_\sigma^+ \mathfrak{u}(t), \quad \widetilde{z}_\sigma^+(0) = P_\sigma^+ \widetilde{z}^0, \quad \mathfrak{B}_\sigma^+ = P_\sigma^+ \mathfrak{B}, \\ &(\widetilde{z}_\sigma^-)'(t) = A_\sigma^- \widetilde{z}_\sigma^-(t) + \mathfrak{B}_\sigma^- \mathfrak{u}(t), \quad \widetilde{z}_\sigma^-(0) = P_\sigma^- \widetilde{z}^0, \quad \mathfrak{B}_\sigma^- = P_\sigma^- \mathfrak{B}, \end{aligned} \tag{2.9}$$

where $\widetilde{z}_\sigma^\pm = P_\sigma^\pm \widetilde{z}$. We need to show that the finite dimensional part $(2.9)_1$ is exactly controllable. Let

$$N_\sigma \geqslant \max_{\operatorname{Re}\lambda_j \geqslant -\sigma} \ell_j, \tag{2.10}$$

where ℓ_j is the geometric multiplicity of the eigenvalue λ_j of the operator A. From [2, Theorem 5] and the first condition in (H1.D), there exists a family $(w_j)_{1\leqslant j\leqslant N_\sigma} \subset \mathbb{U}_\sigma^+ \subset \mathbb{U}$ such that $(2.9)_1$ is exactly controllable. Moreover, it is proved that u is expressed by means of a linear feedback operator

$$u = K_\sigma^+ \widetilde{z}, \quad K_\sigma^+(\cdot) = -\sum_{j=1}^{N_\sigma} \langle w_j, B^*\Pi P_\sigma^+(\cdot)\rangle_{\mathbb{U}} w_j = -\sum_{j=1}^{N_\sigma} (\mathfrak{B}^*(\Pi P_\sigma^+(\cdot)))_j w_j, \tag{2.11}$$

where $\Pi \in \mathcal{L}(\mathbb{H}_\sigma^+, (\mathbb{H}_\sigma^+)^*)$ is the unique solution of the algebraic Riccati equation: for all $\xi, \zeta \in \mathbb{H}_\sigma^+$

$$\begin{aligned} &\langle \xi, \zeta \rangle_{\mathbb{H}} + \langle (A_\sigma^+ + \sigma \operatorname{Id})\xi, \Pi\zeta \rangle_{\mathbb{H}} + \langle \Pi\xi, (A_\sigma^+ + \sigma \operatorname{Id})\zeta \rangle_{\mathbb{H}} \\ &\quad - \textstyle\sum_{j=1}^{N_\sigma} \langle B^*\Pi\xi, w_j \rangle_{\mathbb{U}} \langle B^*\Pi\zeta, w_j \rangle_{\mathbb{U}} = 0, \\ &\langle \Pi\xi, \zeta \rangle_{\mathbb{H}} = \langle \xi, \Pi\zeta \rangle_{\mathbb{H}}, \quad \text{and} \quad \forall \xi \neq 0,\ \langle \Pi\xi, \xi \rangle_{\mathbb{H}} > 0. \end{aligned} \tag{2.12}$$

It is well-known (see for instance [7, page 292]) that this Riccati operator is strongly related to the standard LQR problem with infinite horizon. In particular, this choice ensures that the solution of the finite dimensional system (2.9)$_1$

$$(\widetilde{z}_\sigma^+)'(t) = A_\sigma^+ \widetilde{z}_\sigma^+(t) - \mathfrak{B}_\sigma^+(\mathfrak{B}^*(\Pi P_\sigma^+ \widetilde{z}(t))),$$

is exponentially stable i.e.

$$\|\widetilde{z}_\sigma^+(t)\|_{\mathbb{H}} \leqslant M e^{-(\sigma+\varepsilon)t} \|\widetilde{z}^0\|_{\mathbb{H}}, \quad t > 0.$$

It follows from Duhamel's formula that the whole system (2.9) is exponentially stable (see [2]). We can construct L_σ^+ similarly considering the system

$$\begin{aligned} &\widetilde{z}_\star'(t) = A^* \widetilde{z}_\star(t) + C^* u_\star(t), \\ &\widetilde{z}_\star(0) = \widetilde{z}_\star^0. \end{aligned} \tag{2.13}$$

Using similar arguments and the second condition in (H1.D), we show that there exists a family $(w_j^\star)_{1 \leqslant j \leqslant N_\sigma} \subset \mathbb{Y}_\sigma^+$ such that

$$u_\star = L_\star \widetilde{z}_\star, \quad L_\star(\cdot) = -\sum_{j=1}^{N_\sigma} \langle w_j^\star, C\Pi_\star (P_\sigma^+)^*(\cdot) \rangle_{\mathbb{Y}} w_j^\star = -\sum_{j=1}^{N_\sigma} (\mathfrak{C}_\star^*(\Pi_\star (P_\sigma^+)^*(\cdot)))_j w_j^\star,$$

where

$$\mathfrak{C}_\star : \mathbb{C}^{N_\sigma} \longrightarrow \mathcal{D}(A^*)', \quad \Theta = (\theta_1, \cdots, \theta_{N_\sigma}) \longmapsto \mathfrak{C}_\star \Theta = \sum_{j=1}^{N_\sigma} \theta_j C^* w_j^\star,$$

where $\Pi_\star \in \mathcal{L}((\mathbb{H}_\sigma^+)^*, \mathbb{H}_\sigma^+)$ is the unique solution of the algebraic Riccati equation: for all $\xi, \zeta \in (\mathbb{H}_\sigma^+)^*$

$$\begin{aligned} &\langle \xi, \zeta \rangle_{\mathbb{H}} + \langle ((A_\sigma^+)^* + \sigma \operatorname{Id})\xi, \Pi_\star\zeta \rangle_{\mathbb{H}} + \langle \Pi_\star\xi, ((A_\sigma^+)^* + \sigma \operatorname{Id})\zeta \rangle_{\mathbb{H}} \\ &\quad - \textstyle\sum_{j=1}^{N_\sigma} \langle C\Pi_\star\xi, w_j \rangle_{\mathbb{Y}} \langle C\Pi_\star\zeta, w_j \rangle_{\mathbb{Y}} = 0, \\ &\langle \Pi_\star\xi, \zeta \rangle_{\mathbb{H}} = \langle \xi, \Pi_\star\zeta \rangle_{\mathbb{H}}, \quad \text{and} \quad \forall \xi \neq 0,\ \langle \Pi_\star\xi, \xi \rangle_{\mathbb{H}} > 0. \end{aligned} \tag{2.14}$$

Hence, we define

$$L_\sigma^+(\cdot) = L_\star^*(\cdot) = -\sum_{j=1}^{N_\sigma} \langle w_j^\star, \cdot \rangle_{\mathbb{Y}} \chi_j, \tag{2.15}$$

with

$$\chi_j = P_\sigma^+ \Pi_\star C^* w_j^\star \in \mathbb{H}_\sigma^+. \tag{2.16}$$

With this choice, we get that $(A + L_\sigma^+ C)^*$ and hence $A + L_\sigma^+ Q_\sigma^+ C$ are exponentially stable with decay rate less than $-\sigma$. Finally, using K_σ^+ and L_σ^+ we construct the observer $\widehat{z}$ satisfying (1.6–1.7), that is

$$\begin{aligned} &\widehat{z}'(t) = A\widehat{z}(t) + BK_\sigma^+ \widehat{z}(t) + L_\sigma^+ Q_\sigma^+ (C\widehat{z}(t) - y(t)), \\ &\widehat{z}(0) = 0. \end{aligned} \tag{2.17}$$

2.3 Stability of the Closed-Loop System

We define the error $e = z - \widehat{z}$. Then, we obtain

$$\begin{aligned} e'(t) = (A + L_\sigma^+ Q_\sigma^+ C)e(t), \qquad & e(0) = z^0, \\ z'(t) = (A + BK_\sigma^+)z(t) - BK_\sigma^+ e(t), \; & z(0) = z^0. \end{aligned} \tag{2.18}$$

We prove that e is exponentially stable with decay rate $-\sigma$.

Proposition 2.2 *Systems (2.17) and (2.18) are exponentially stable with decay rate $-\sigma$.*

Proof Since $(A + L_\sigma^+ Q_\sigma^+ C)$ is of negative type strictly less than $-\sigma$, then there exists $0 < \varepsilon'' < \varepsilon$ such that

$$\|e(t)\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon'')} \|z^0\|_{\mathbb{H}}. \tag{2.19}$$

Going back to system (2.5) with the control given by (1.6), we have, since $K_\sigma^+ z_\sigma^- = 0$,

$$(z_\sigma^+)'(t) = (A_\sigma^+ + B_\sigma^+ K_\sigma^+) z_\sigma^+(t) - B_\sigma^+ K_\sigma^+ e(t), \quad z_\sigma^+(0) = P_\sigma^+ z^0.$$

Moreover, there exists $\varepsilon' > 0$ with $\varepsilon'' < \varepsilon' < \varepsilon$ such that $(A_\sigma^+ + B_\sigma^+ K_\sigma^+)$ is exponentially stable with rate $-\sigma - \varepsilon'$. We have

$$z_\sigma^+(t) = e^{t(A_\sigma^+ + B_\sigma^+ K_\sigma^+)} P_\sigma^+ z^0 - \int_0^t e^{(t-s)(A_\sigma^+ + B_\sigma^+ K_\sigma^+)} B_\sigma^+ K_\sigma^+ e(s) ds.$$

From (2.19), we see that

$$\|z_\sigma^+(t)\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon'')} \|z^0\|_{\mathbb{H}}. \tag{2.20}$$

We deal now with the infinite dimensional part z_σ^- of the state. From (2.5) with the control given by (1.6), we have

$$(z_\sigma^-)'(t) = A_\sigma^- z_\sigma^-(t) + B_\sigma^- p_\sigma^- K_\sigma^+ z_\sigma^+(t) - B_\sigma^- p_\sigma^- K_\sigma^+ e(t), \quad z_\sigma^-(0) = (\mathrm{Id} - P_\sigma^+)z^0.$$

Using Duhamel's formula, we obtain that

$$z_\sigma^-(t) = e^{tA_\sigma^-}(\mathrm{Id} - P_\sigma^+)z^0 + \int_0^t e^{(t-s)A_\sigma^-} B_\sigma^- p_\sigma^- K_\sigma^+ \left(z_\sigma^+(s) - e(s)\right)\, ds.$$

We note that since the resolvent commutes with the projection P_σ^+ and $e^{(t-s)A_\sigma^-}$, we obtain for $\mu_0 \in \rho(A)$ and $\gamma \in [0, 1)$,

$$\begin{aligned} e^{(t-s)A_\sigma^-} B_\sigma^- &= e^{(t-s)A_\sigma^-}(\mu_0 \mathrm{Id} - A)^\gamma (\mathrm{Id} - P_\sigma^+)(\mu_0 \mathrm{Id} - A)^{-\gamma} B i_\sigma^- \\ &= e^{(t-s)A_\sigma^-}(\mu_0 \mathrm{Id} - A_\sigma^-)^\gamma (\mathrm{Id} - P_\sigma^+)(\mu_0 \mathrm{Id} - A)^{-\gamma} B i_\sigma^- \\ &= (\mu_0 \mathrm{Id} - A_\sigma^-)^\gamma e^{(t-s)A_\sigma^-} (\mathrm{Id} - P_\sigma^+)(\mu_0 \mathrm{Id} - A)^{-\gamma} B i_\sigma^-. \end{aligned} \tag{2.21}$$

Using (H1.B), (2.3), (2.19), (2.20) and (2.21), we get

$$\begin{aligned} \|z_\sigma^-(t)\|_{\mathbb{H}} &\leqslant M\|z^0\|_{\mathbb{H}} \left(e^{-t(\sigma+\varepsilon)} + \int_0^t \frac{1}{(t-s)^\gamma} e^{-(t-s)(\sigma+\varepsilon)} e^{-s(\sigma+\varepsilon'')}\, ds\right) \\ &\leqslant M e^{-t(\sigma+\varepsilon'')} \|z^0\|_{\mathbb{H}} \left(1 + \int_0^t \frac{1}{(t-s)^\gamma} e^{(t-s)(\varepsilon''-\varepsilon)}\, ds\right). \end{aligned} \tag{2.22}$$

Then, since $\varepsilon'' < \varepsilon$, we obtain

$$\|z_\sigma^-(t)\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon'')} \|z^0\|_{\mathbb{H}}. \tag{2.23}$$

Then from (2.19), (2.20) and (2.23), we obtain that z, $\widehat{z}$ and the error e are exponentially stable. □

That concludes the proof of Theorem 1.1.

Remark 2.3 According to (1.6) and (2.11), the control reads

$$v(t) = \sum_{i=1}^{N_\sigma} K_i\left(\widehat{z_\sigma^+}\right) w_i,$$

with $K_i \in \mathcal{L}(\mathbb{H}_\sigma^+, \mathbb{C})$ and $w_i \in \mathbb{U}_\sigma^+$, $i = 1, \ldots, N_\sigma$. From the decomposition (2.2) and the fact that $(\mathbb{H}_\sigma^+)^\perp = (\mathbb{H}_\sigma^-)^*$, we have that if $\zeta \in (\mathbb{H}_\sigma^+)^*$, then

$$\forall \phi \in \mathbb{H}_\sigma^+, \quad \langle \phi, \zeta \rangle_{\mathbb{H}} = 0 \implies \zeta = 0.$$

Since $\dim\left((\mathbb{H}_\sigma^+)^*\right) = \dim \mathcal{L}(\mathbb{H}_\sigma^+, \mathbb{C})$, we infer that there exists a unique $\zeta_i \in (\mathbb{H}_\sigma^+)^* \subset \mathcal{D}(A^*)$ such that

$$K_i\left(\widehat{z_\sigma^+}\right) = \langle \widehat{z}, \zeta_i \rangle_{\mathbb{H}}.$$

In other words, the control can also be written in the form

$$v(t) = \sum_{i=1}^{N_\sigma} \langle \widehat{z}, \zeta_i \rangle_{\mathbb{H}} \, w_i .$$

In the special case where there is only one unstable simple eigenvalue (with an eigenspace spanned by an eigenfunction $\varepsilon_1 \in \mathbb{H}$), the above relations take simpler forms. Indeed, we have then

$$w_1 = B^* \varepsilon_1,$$

and for all $\varphi \in \mathbb{H}_\sigma^+ = \mathrm{Span}\{\varepsilon_1\}$:

$$K_1(\varphi) = -\langle B^*\varepsilon_1, B^*\Pi P_\sigma^+ \varphi\rangle_{\mathbb{U}} = \langle \zeta_1, \varphi\rangle_{(\mathbb{H}_\sigma^+)^*, \mathbb{H}_\sigma^+}, \qquad \zeta_1 = -(P_\sigma^+)^* \Pi B B^* \varepsilon_1 .$$

3 Finite Dimensional Observer

3.1 Spectral Decomposition of the System

In this section, we assume hypotheses (H2.A), (H2.B), (H2.C) and (H2.D) to hold true.

Consider $\nu > 0$ and let us introduce the projection operators P_ν^+ as in (1.5) where in this case Γ_ν^+ is a circle enclosing Σ_ν^+ but no other point of the spectrum of A and oriented counterclockwise (see [12, V.5, p.272]). Since A is a self-adjoint operator, then P_ν^+ is well defined. Moreover from the expression of the projections, it follows that

$$(P_\nu^+)^* = P_\nu^+ .$$

Thus, P_ν^+ is orthogonal projection of norm equal to 1. We set

$$\mathbb{H}_\nu^+ = P_\nu^+ \mathbb{H}, \qquad \mathbb{H}_\nu^- = (\mathrm{Id} - P_\nu^+)\mathbb{H}, \qquad \mathbb{H} = \mathbb{H}_\nu^+ \oplus \mathbb{H}_\nu^-,$$

and

$$A_\nu^+ := A_{|\mathbb{H}_\nu^+} : \mathbb{H}_\nu^+ \to \mathbb{H}_\nu^+, \qquad A_\nu^- := A_{|\mathbb{H}_\nu^-} : \mathcal{D}(A) \cap \mathbb{H}_\nu^- \to \mathbb{H}_\nu^- .$$

We also define as before

$$\mathbb{U}_\nu^+ := B^* \mathbb{H}_\nu^+, \qquad \mathbb{U}_\nu^- := B^* \left(\mathcal{D}(A) \cap \mathbb{H}_\nu^- \right),$$

and

$$p_\nu^+ : \mathbb{U} \to \mathbb{U}_\nu^+, \quad p_\nu^- : \mathbb{U} \to \mathbb{U}_\nu^-, \quad i_\nu^+ : \mathbb{U}_\nu^+ \to \mathbb{U}, \quad i_\nu^- : \mathbb{U}_\nu^- \to \mathbb{U},$$

the orthogonal projections and the inclusion maps. Note that we have the following relations for the above maps:

$$i_\nu^+ = (p_\nu^+)^*, \qquad i_\nu^- = (p_\nu^-)^*. \tag{3.1}$$

We can thus define

$$B_\nu^+ := P_\nu^+ B i_\nu^+ \in \mathcal{L}(\mathbb{U}_\nu^+, \mathbb{H}_\nu^+), \quad B_\nu^- := (\mathrm{Id} - P_\nu^+) B i_\nu^- \in \mathcal{L}(\mathbb{U}_\nu^-, \mathbb{H}_\nu^-).$$

It is proved in [2] (see also [3] and [20]) that

$$P_\nu^+ B = B_\nu^+ p_\nu^+, \qquad (\mathrm{Id} - P_\nu^+) B = B_\nu^- p_\nu^-.$$

We introduce also the orthogonal projection Q_ν^+ from $\mathbb{Y}$ into $\mathbb{Y}_\nu^+ = C\mathbb{H}_\nu^+$ and define

$$C_\nu^+ = C\iota_\nu^+,$$

where ι_ν^+ designates the injection operator from $\mathbb{H}_\nu^+$ to $\mathbb{H}$. Consider now $\sigma^\star > \sigma > 0$. We take $\nu = \sigma$ or $\nu = \sigma^\star$ in the maps and spaces defined previously. Since A is self-adjoint with compact resolvent, we deduce the existence of $\varepsilon > 0$ such that for all $t \geqslant 0$

$$\|e^{A_\sigma^- t}\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant e^{-(\sigma+\varepsilon)t}, \tag{3.2}$$

and

$$\|e^{A_{\sigma^\star}^- t}\|_{\mathcal{L}(\mathbb{H}_{\sigma^\star}^-)} \leqslant e^{-\sigma^\star t}. \tag{3.3}$$

The system (1.3) splits into

$$(z_{\sigma^\star}^+)'(t) = A_{\sigma^\star}^+ z_{\sigma^\star}^+(t) + B_{\sigma^\star}^+ p_{\sigma^\star}^+ v(t), \quad z_{\sigma^\star}^+(0) = P_{\sigma^\star}^+ z^0, \tag{3.4}$$

$$(z_{\sigma^\star}^-)'(t) = A_{\sigma^\star}^- z_{\sigma^\star}^-(t) + B_{\sigma^\star}^- p_{\sigma^\star}^- v(t), \quad z_{\sigma^\star}^-(0) = (\mathrm{Id} - P_{\sigma^\star}^+) z^0. \tag{3.5}$$

3.2 Finite Dimensional Observer-Based Control and Stability of the Closed-Loop System

We are now in position to prove Theorem 1.2. The matrices K_σ^+ and L_σ^+ being respectively given by (2.11) and (2.15), we define the finite dimensional observer-based feedback control by

$$v(t) = K_\sigma^+ \widehat{z}_\star(t) \in B^* P_\sigma^+ \mathbb{H} \subset B^* P_{\sigma^\star}^+ \mathbb{H}, \tag{3.6}$$

where the finite dimensional observer $\widehat{z}_\star \in P_{\sigma^\star}^+ \mathbb{H}$ solves

$$\begin{aligned} &\widehat{z}_\star'(t) = A_{\sigma^\star}^+ \widehat{z}_\star(t) + B_{\sigma^\star}^+ v(t) + L_\sigma^+ Q_\sigma^+ (C_{\sigma^\star}^+ \widehat{z}_\star(t) - y(t)), \\ &\widehat{z}_\star(0) = 0. \end{aligned} \tag{3.7}$$

Our goal is to prove that the coupled system (1.3), (3.6) and (3.7) is exponentially stable.

We define the error $e = z^+_{\sigma^\star} - \widehat{z}_\star$ which satisfies the following system

$$\begin{aligned} &e'(t) = A^+_{\sigma^\star} e(t) + L^+_\sigma Q^+_\sigma (C^+_{\sigma^\star} e(t) + C z^-_{\sigma^\star}(t)), \\ &e(0) = z^+_{\sigma^\star}(0). \end{aligned} \tag{3.8}$$

We prove that $\widehat{z}$ and e are exponentially stable with decay rate $-\sigma$. Let us set

$$\mathbb{X} = \begin{pmatrix} \widehat{z}_\star \\ e \end{pmatrix}, \qquad \mathbb{X}^0 = \begin{pmatrix} 0 \\ z^+_{\sigma^\star}(0) \end{pmatrix}.$$

Using (3.7) and (3.8), $\mathbb{X}$ satisfies the system

$$\begin{aligned} &\mathbb{X}'(t) = \mathbb{A}\mathbb{X}(t) + \mathbb{L}(z^-_{\sigma^\star}(t)), \\ &\mathbb{X}(0) = \mathbb{X}^0, \end{aligned} \tag{3.9}$$

where

$$\mathbb{A} = \begin{pmatrix} A^+_{\sigma^\star} + B^+_{\sigma^\star} K^+_\sigma & -L^+_\sigma Q^+_\sigma C^+_{\sigma^\star} \\ 0 & A^+_{\sigma^\star} + L^+_\sigma Q^+_\sigma C^+_{\sigma^\star} \end{pmatrix}, \quad \mathbb{L}(z^-_{\sigma^\star}) = \begin{pmatrix} -L^+_\sigma Q^+_\sigma C z^-_{\sigma^\star} \\ L^+_\sigma Q^+_\sigma C z^-_{\sigma^\star} \end{pmatrix}. \tag{3.10}$$

Let us first prove that $\mathbb{A}$ is stable matrix. In the sequel, the constant M is a generic constant that can change from a line to another but need to be independent of $\sigma^\star$.

Lemma 3.1 *The matrices $A^+_{\sigma^\star} + B^+_{\sigma^\star} K^+_\sigma$ and $A^+_{\sigma^\star} + L^+_\sigma Q^+_\sigma C^+_{\sigma^\star}$ are exponentially stable with a decay rate less than $-\sigma$.*

Proof Let $\xi^0 \in \mathbb{H}^+_{\sigma^\star}$ be given. To prove that $A^+_{\sigma^\star} + B^+_{\sigma^\star} K^+_\sigma$ is exponentially stable, we only need to show that the solution $\xi(t)$ of the finite dimensional system

$$\begin{aligned} &\xi'(t) = (A^+_{\sigma^\star} + B^+_{\sigma^\star} K^+_\sigma)\xi(t), \\ &\xi(0) = \xi^0, \end{aligned} \tag{3.11}$$

is exponentially decaying. Consider then the infinite dimensional system

$$\begin{aligned} &x'(t) = (A + BK^+_\sigma)x(t), \\ &x(0) = \xi^0. \end{aligned} \tag{3.12}$$

From Section 2.2, we see that the system (3.12) is exponentially stable of decay rate $-\sigma - \varepsilon'$. It implies that

$$\|x(t)\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon')} \|\xi^0\|_{\mathbb{H}}. \tag{3.13}$$

On the other hand, applying $P^+_{\sigma^\star}$ to (3.12) and recalling that $K^+_\sigma x(t)$ acts only on the projected part of $x(t)$ on $\mathbb{H}^+_\sigma$, we obtain that

$$\begin{aligned}&(x^+_{\sigma^\star})'(t) = A^+_{\sigma^\star} x^+_{\sigma^\star}(t) + B^+_{\sigma^\star} K^+_\sigma x(t) = (A^+_{\sigma^\star} + B^+_{\sigma^\star} K^+_\sigma) x^+_{\sigma^\star}(t),\\ &x^+_{\sigma^\star}(0) = \xi^0.\end{aligned} \tag{3.14}$$

This shows that $x^+_{\sigma^\star}(t)$ is the unique solution $\xi(t)$ of (3.11), and we get from (3.13), that

$$\|\xi(t)\|_{\mathbb{H}} = \|x^+_{\sigma^\star}(t)\|_{\mathbb{H}} = \|P^+_{\sigma^\star} x(t)\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon')} \|\xi^0\|_{\mathbb{H}},$$

for all $\xi^0 \in \mathbb{H}^+_{\sigma^\star}$. Hence

$$\left\| e^{t(A^+_{\sigma^\star} + B^+_{\sigma^\star} K^+_\sigma)} \xi^0 \right\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon')} \|\xi^0\|_{\mathbb{H}},$$

and the matrix $A^+_{\sigma^\star} + B^+_{\sigma^\star} K^+_\sigma$ is exponentially stable with a decay rate less than $-\sigma$. We use the same argument for $A^+_{\sigma^\star} + L^+_\sigma Q^+_\sigma C^+_{\sigma^\star}$ by considering its adjoint $(A^+_{\sigma^\star})^* + (C^+_{\sigma^\star})^*(L^+_\sigma)^*$ that has exactly the same form as the one previously studied. □

Since $\mathbb{A}$ is a triangular matrix, using Lemma 3.1 and Duhamel's formula, we obtain that $\mathbb{A}$ is stable with exponential rate strictly less than $-\sigma$. We can now prove the exponential stability of the full closed-loop system (3.7) and (1.3):

$$\begin{aligned}\mathbb{X}'(t) &= \mathbb{A}\mathbb{X}(t) + \mathbb{L}(z^-_{\sigma^\star}(t)),\\ (z^-_{\sigma^\star})'(t) &= A^-_{\sigma^\star} z^-_{\sigma^\star}(t) + B^-_{\sigma^\star} p^-_{\sigma^\star} \mathbb{K}^+_\sigma \mathbb{X}(t),\end{aligned} \tag{3.15}$$

where $\mathbb{K}^+_\sigma = (K^+_\sigma, 0)$ and with the initial conditions

$$\begin{aligned}\mathbb{X}(0) &= \mathbb{X}^0,\\ z^-_{\sigma^\star}(0) &= (\mathrm{Id} - P^+_{\sigma^\star}) z^0.\end{aligned}$$

From Duhamel's formula, the two first equations in (3.15) also read

$$\begin{aligned}\mathbb{X}(s) &= e^{s\mathbb{A}}\mathbb{X}(0) + \int_0^s e^{(s-\tau)\mathbb{A}} \mathbb{L}(z^-_{\sigma^\star}(\tau)) d\tau,\\ z^-_{\sigma^\star}(t) &= e^{tA^-_{\sigma^\star}} z^-_{\sigma^\star}(0) + \int_0^t e^{(t-s)A^-_{\sigma^\star}} \left(B^-_{\sigma^\star} p^-_{\sigma^\star} \mathbb{K}^+_\sigma \mathbb{X}(s)\right) ds.\end{aligned}$$

Substituting the first equation above into the second one yields

$$\begin{aligned}z^-_{\sigma^\star}(t) &= e^{tA^-_{\sigma^\star}} z^-_{\sigma^\star}(0)\\ &\quad + \int_0^t e^{(t-s)A^-_{\sigma^\star}} B^-_{\sigma^\star} p^-_{\sigma^\star} \mathbb{K}^+_\sigma \left(e^{s\mathbb{A}}\mathbb{X}(0) + \int_0^s e^{(s-\tau)\mathbb{A}} \mathbb{L}(z^-_{\sigma^\star}(\tau)) d\tau \right) ds.\end{aligned} \tag{3.16}$$

Setting

$$Z^0 := e^{tA^-_{\sigma^\star}} z^-_{\sigma^\star}(0) + \int_0^t e^{(t-s)A^-_{\sigma^\star}} \left(B^-_{\sigma^\star} p^-_{\sigma^\star} \mathbb{K}^+_\sigma \left(e^{s\mathbb{A}} \mathbb{X}(0) \right) \right) ds,$$

relation (3.16) can be written

$$z^-_{\sigma^\star}(t) = \int_0^t \int_0^s e^{(t-s)A^-_{\sigma^\star}} B^-_{\sigma^\star} p^-_{\sigma^\star} \mathbb{K}^+_\sigma \left(e^{(s-\tau)\mathbb{A}} \mathbb{L}(z^-_{\sigma^\star}(\tau)) d\tau ds \right) + Z^0. \quad (3.17)$$

To prove the stability of $z^-_{\sigma^\star}$, we prove the existence of a unique solution to (3.17) in a weighted space by using a fixed point argument. More precisely, let us define the following map

$$\begin{aligned} \Phi : L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star}) &\longrightarrow L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star}) \\ g &\longmapsto \Phi(g) := \int_0^t \int_0^s e^{(t-s)A^-_{\sigma^\star}} B^-_{\sigma^\star} p^-_{\sigma^\star} \mathbb{K}^+_\sigma e^{(s-\tau)\mathbb{A}} \mathbb{L}(g(\tau))\, d\tau ds + Z^0. \end{aligned}$$

Then, equation (3.16) simply reads

$$\Phi(z^-_{\sigma^\star}) = z^-_{\sigma^\star}.$$

First, we prove that the function Φ is well defined. Given $g \in L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})$. Since C is bounded, $\mathbb{L}$ is also bounded. Hence, using the definition of Φ, (H2.B) (boundedness of B), (3.3) (exponential decay of $A^-_{\sigma^\star}$) and the exponential decay of $\mathbb{A}$, we obtain that there exists $\varepsilon'' > 0$ such that

$$\begin{aligned} \|\Phi(g)(t)\|_{\mathbb{H}} \leqslant M \int_0^t \int_0^s e^{-(t-s)\sigma^\star} e^{-(s-\tau)(\sigma+\varepsilon")} e^{-\tau\sigma} \|e^{\tau\sigma} g(\tau)\|_{\mathbb{H}}\, d\tau\, ds \\ + M \left\| z^0 \right\|_{\mathbb{H}} \int_0^t e^{-(t-s)\sigma^\star} e^{-s\sigma}\, ds + e^{-t\sigma^\star} \|z^0\|_{\mathbb{H}}. \end{aligned}$$

Consequently

$$\|\Phi(g)(t)\|_{\mathbb{H}} \leqslant I_1 + I_2 + e^{-t\sigma^\star} \|z^0\|_{\mathbb{H}}, \quad (3.18)$$

where we have set

$$\begin{aligned} I_1 := M \int_0^t e^{-(t-s)\sigma^\star} e^{-s\sigma} \left(\int_0^s e^{-(s-\tau)\varepsilon''} \|e^{\tau\sigma} g(\tau)\|_{\mathbb{H}}\, d\tau \right) ds, \\ I_2 := M \left\| z^0 \right\|_{\mathbb{H}} \int_0^t e^{-(t-s)\sigma^\star} e^{-s\sigma}\, ds. \end{aligned}$$

Noticing that

$$\int_0^s e^{-(s-\tau)\varepsilon''}\, d\tau = \frac{1}{\varepsilon''} \left[1 - e^{-s\varepsilon''} \right] \leqslant \frac{1}{\varepsilon''},$$

we get that

$$\int_0^s e^{-(s-\tau)\varepsilon''} \|e^{\tau\sigma} g(\tau)\|_{\mathbb{H}}\, d\tau \leqslant \frac{1}{\varepsilon''} \|g\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma*})},$$

and hence

$$I_1 \leqslant M \|g\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma*})} \frac{e^{-t\sigma}}{\varepsilon''} \int_0^t e^{(t-s)(\sigma-\sigma^\star)}\, ds.$$

Since

$$\int_0^t e^{(t-s)(\sigma-\sigma^\star)}\, ds = \frac{1-e^{-t(\sigma^*-\sigma)}}{(\sigma^\star-\sigma)} \leqslant \frac{1}{(\sigma^\star-\sigma)},$$

we have

$$I_1 \leqslant \frac{Me^{-\sigma t}}{\varepsilon''(\sigma^\star-\sigma)} \|g\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})},$$

and similarly

$$I_2 \leqslant Me^{-\sigma t}\left(\int_0^t e^{(t-s)(\sigma-\sigma^\star)}\, ds\right)\left\|z^0\right\|_{\mathbb{H}} \leqslant \frac{Me^{-\sigma t}}{(\sigma^\star-\sigma)}\left\|z^0\right\|_{\mathbb{H}}.$$

Using the above estimates in (3.18), we get that

$$\|\Phi(g)(t)\|_{\mathbb{H}} \leqslant Me^{-\sigma t}\left(\frac{1}{\varepsilon''(\sigma^\star-\sigma)}\|g\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})} + \left\|z^0\right\|_{\mathbb{H}} + \frac{1}{\sigma^*-\sigma}\left\|z^0\right\|_{\mathbb{H}}\right),$$

and hence $\Phi(g) \in L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})$.

It remains to show that Φ is a contraction mapping. Given $g_1, g_2 \in L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})$, the same calculations as above show that

$$\|\Phi(g_1)-\Phi(g_2)\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})} \leqslant \frac{M}{\varepsilon''(\sigma^\star-\sigma)}\|g_1-g_2\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})}.$$

The application Φ is thus a contraction provided that $\sigma^\star$ is chosen large enough to ensure that

$$\frac{M}{\varepsilon''(\sigma^\star-\sigma)} < 1. \tag{3.19}$$

Then, applying the fixed point theorem we get that there exists a unique $z^-_{\sigma^\star} \in L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})$ such that $\Phi(z^-_{\sigma^\star}) = z^-_{\sigma^\star}$ and

$$\|z^-_{\sigma^\star}(t)\|_{\mathbb{H}} \leqslant Me^{-\sigma t}\left\|z^0\right\|_{\mathbb{H}}.$$

Moreover, going back to the first equation in (3.15), and using Duhamel's formula again, we easily obtain that

$$\|\mathbb{X}(t)\|_{\mathbb{H}\times\mathbb{H}} \leqslant Me^{-\sigma t}\left\|z^0\right\|_{\mathbb{H}}.$$

This completes the proof of Theorem 1.2.

4 Stabilization of the Reaction–Diffusion Equation by an Infinite Dimensional Observer

In this section, we apply Theorem 1.1 for the stabilization of the heat equation. Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ be a bounded domain with smooth boundary. Let us consider Γ a non-empty open subset of $\partial\Omega$ and the control problem:

$$\begin{aligned} \partial_t z(t,x) - \Delta z(t,x) - cz(t,x) &= 0 \text{ in } (0,\infty)\times\Omega, \\ z(t,x) &= v(t,x) \text{ on } (0,\infty)\times\Gamma, \\ z(t,x) &= 0 \text{ on } (0,\infty)\times(\partial\Omega\setminus\Gamma), \\ z(0,\cdot) &= z^0 \text{ in } \Omega, \\ y(t,x) &= \mathbb{1}_{\mathcal{O}} z(t,x) \text{ in } (0,\infty)\times\Omega, \end{aligned} \tag{4.1}$$

where $c \in L^\infty(\Omega)$, $\mathcal{O}$ a non empty open subset of $\mathbb{R}^N$, with $\overline{\mathcal{O}} \subset \Omega$. We assume that $z^0 \in L^2(\Omega)$.

In order to write (4.1) under the form (1.3), we introduce the following functional setting:

$$\begin{aligned} \mathbb{H} = L^2(\Omega), \quad \mathbb{U} = L^2(\Gamma), \\ Az = \Delta z + cz, \quad \mathcal{D}(A) = H^2(\Omega)\cap H_0^1(\Omega), \end{aligned}$$

$$\mathbb{Y} = L^2(\mathcal{O}), \quad C = \mathbb{1}_{\mathcal{O}}, \quad C \in \mathcal{L}(\mathbb{H},\mathbb{Y}).$$

The operator $(A,\mathcal{D}(A))$ generates an analytical semigroup with compact resolvent, this is a direct consequence of [4, Theorem 2.12, p.115]. Thus (H1.A) is satisfied.

To define the control operator B, we use a standard method (see, for instance [24, pp.341-343] or [19]): we first consider the lifting operator $D_0 \in \mathcal{L}(L^2(\partial\Omega); L^2(\Omega))$ such that for any $v \in L^2(\partial\Omega)$, $w = D_0 v$ is the unique solution of the following system

$$\begin{aligned} \mu_0 w - \Delta w - cw &= 0 \text{ in } \Omega, \\ w &= v \text{ on } \partial\Omega, \end{aligned}$$

where $\mu_0 \in \rho(A)$. Let us recall how we can reduce (4.1) to an evolution problem (1.3). We set $\widetilde{z} = z - w$, with $w = D_0 v$. Then $\widetilde{z}$ is the weak solution of the system

$$\begin{aligned} \partial_t \widetilde{z} - \Delta \widetilde{z} - c\widetilde{z} = -\partial_t w + \mu_0 w \text{ in } (0,\infty) \times \Omega, \\ \widetilde{z} = 0 \text{ on } (0,\infty) \times \partial\Omega, \\ \widetilde{z}(0,\cdot) = \widetilde{z}^0 := z^0 - w(0,\cdot) \ \text{ in } \ \Omega. \end{aligned}$$

Using Duhamel's formula, we have

$$\widetilde{z}(t) = e^{tA}\widetilde{z}^0 + \int_0^t e^{(t-s)A}(-\partial_t w(s) + \mu_0 w(s))\, ds.$$

By integrating by parts, we obtain

$$z(t) = e^{tA} z^0 + \int_0^t e^{(t-s)A}(\mu_0 \operatorname{Id} - A) w(s)\, ds. \tag{4.2}$$

Remark 4.1 The above computations are valid only for smooth solutions. It is worth noticing that formula (4.2) makes sense for $v \in L^2(\partial\Omega)$. In fact, to handle the general case, we refer to [24, pp.342] and the definition of weak solutions therein.

We get finally that the problem (4.1) can be rewritten as

$$\begin{aligned} & z'(t) = Az(t) + Bv(t), \quad t > 0,\ z(0) = z^0, \\ & y(t) = Cz(t), \end{aligned}$$

with

$$B = (\mu_0 \operatorname{Id} - A) D_0 : \mathbb{U} \longrightarrow (\mathcal{D}(A^*))',$$

where we have extended the operator A as an operator from $L^2(\Omega)$ into $(D(A^*))'$ and where we see $\mathbb{U}$ as a closed subspace of $L^2(\partial\Omega)$ (by extending by zero in $\partial\Omega \setminus \Gamma$ any $v \in \mathbb{U}$). Using standard results on elliptic equations, we have that B satisfies (H1.B) for any $\gamma > 3/4$ (see for instance [18, Theorem 2.6]). To apply Theorem 1.1, we only need to check (H1.D). We recall that

$$\mathcal{D}(A^*) = H^2(\Omega) \cap H_0^1(\Omega), \quad A^* = A.$$

Moreover, by classical results (see [24, Proposition 10.6.7]), we see that

$$D_0^* := -\frac{\partial}{\partial \nu}(\mu_0 \operatorname{Id} - A^*)^{-1} = -\frac{\partial}{\partial \nu}(\mu_0 \operatorname{Id} - A)^{-1},$$

and thus

$$B^* \varepsilon := -\frac{\partial \varepsilon}{\partial \nu}_{|\Gamma}.$$

Thus if ε satisfies $A^*\varepsilon = \lambda\varepsilon$ and $B^*\varepsilon = 0$, then

$$\begin{aligned} \lambda\varepsilon - \Delta\varepsilon - c\varepsilon &= 0 \text{ in } \Omega, \\ \varepsilon &= 0 \text{ on } \partial\Omega, \\ \frac{\partial\varepsilon}{\partial\nu} &= 0 \text{ on } \Gamma. \end{aligned}$$

The unique continuation property is a consequence of Holmgren's uniqueness theorem (see [11, Theorem 3.5.1, p.125]), by using an extension of the domain. On the other hand, if ε satisfies $A\varepsilon = \lambda\varepsilon$ and $C\varepsilon = 0$, then

$$\begin{aligned} \lambda\varepsilon - \Delta\varepsilon - c\varepsilon &= 0 \text{ in } \Omega, \\ \varepsilon &= 0 \text{ on } \partial\Omega, \\ \varepsilon &= 0 \text{ in } \mathcal{O}. \end{aligned}$$

Applying Holmgren's theorem again, we obtain $\varepsilon = 0$. Thus (H1.D) holds for any σ.

Now, we define the observer $\widehat{z}$ using Remark 2.3. Let us define N_σ by (2.10) and $\widehat{z}$ the solution of the closed loop system

$$\begin{array}{rll} \partial_t\widehat{z} - \Delta\widehat{z} - c\widehat{z} = \sum_{i=1}^{N_\sigma} \left\langle \mathbb{1}_{\mathcal{O}}(\widehat{z} - z), w_i^\star \right\rangle_{\mathbb{Y}} \chi_i & \text{in } (0,\infty)\times\Omega, \\ \widehat{z} = \sum_{i=1}^{N_\sigma} \langle \widehat{z}, \zeta_i \rangle_{\mathbb{H}}\, w_i & \text{on } (0,\infty)\times\Gamma, \\ \widehat{z} = 0 & \text{on } (0,\infty)\times(\partial\Omega\setminus\Gamma), \\ \widehat{z}(0,\cdot) = 0 & \text{in } \Omega, \end{array} \tag{4.3}$$

where $(w_i^\star)_{1\leqslant i\leqslant N_\sigma} \subset \mathbb{Y}$, $(\chi_i)_{1\leqslant i\leqslant N_\sigma} \subset \mathbb{H}$, $(\zeta_i)_{1\leqslant i\leqslant N_\sigma} \subset \mathcal{D}(A^*)$ and $(w_i)_{1\leqslant i\leqslant N_\sigma} \subset \mathbb{U}$ (see Eqs. (2.15–2.16) and Remark 2.3). We deduce the following result by applying Theorem 1.1:

Theorem 4.2 *Assume $\sigma > 0$. There exists a control*

$$v(t) = \sum_{i=1}^{N_\sigma} \left(\int_\Omega \widehat{z}(t)\zeta_i\, dx \right) w_i, \tag{4.4}$$

with $\zeta_i \in H^2(\Omega)\cap H_0^1(\Omega)$, $w_i \in H^{1/2}(\Gamma)$, $i = 1,\ldots,N_\sigma$ such that the coupled system (4.1) and (4.3) is exponentially stable that satisfies for $z^0 \in L^2(\Omega)$ the estimate

$$\|z(t)\|_{L^2(\Omega)} \leqslant Ce^{-\sigma t}\|z^0\|_{L^2(\Omega)}. \tag{4.5}$$

Remark 4.3 In dimension 2, we can obtain the same result for the operator

$$Az = \nabla\cdot(b\nabla z) + cz, \quad \mathcal{D}(A) = H^2(\Omega)\cap H_0^1(\Omega),$$

with b a positive-definite symmetric matrix of variable coefficients in $L^\infty(\Omega)$, using the unique continuation property from [1, Theorem 3.2].

5 Stabilization of the Reaction–Diffusion Equation by a Finite Dimensional Observer

Let Ω be a domain of $\mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary. In this section, we apply Theorem 1.2 to the following initial boundary value problem

$$\begin{aligned} \partial_t z(t,x) - \Delta z(t,x) - cz(t,x) &= \mathbb{1}_{\mathcal{O}'} v(t,x) && \text{in } (0,\infty)\times\Omega,\\ z(t,x) &= 0 && \text{on } (0,\infty)\times\partial\Omega,\\ z(0,\cdot) &= z^0 && \text{in } \Omega,\\ y(t,x) &= \mathbb{1}_{\mathcal{O}} z(t,x) && \text{in } (0,\infty)\times\Omega, \end{aligned} \tag{5.1}$$

where $\mathcal{O}$ and $\mathcal{O}'$ are two non empty open sets such that $\overline{\mathcal{O}} \subset \Omega$ and $\overline{\mathcal{O}'} \subset \Omega$. In this case, we have

$$\mathbb{H} = L^2(\Omega), \quad \mathbb{U} = L^2(\mathcal{O}'),$$
$$Az = \Delta z + cz, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

The operator A is self-adjoint with compact resolvent, then in particular, we see that (H2.A) holds true. Moreover,

$$B : \mathbb{U} \to \mathbb{H}, \quad B = \mathbb{1}_{\mathcal{O}'},$$

and

$$\mathbb{Y} = L^2(\mathcal{O}), \quad C : \mathbb{H} \to \mathbb{Y}, \quad C = \mathbb{1}_{\mathcal{O}}.$$

We can check as the previous section that (H2.B), (H2.C) and (H2.D) are verified. Let fix $\sigma > 0$ and $\sigma^\star > \sigma$. The hypothesis (H2.A) implies in particular that A is diagonalizable. Hence

$$\mathbb{H}_\sigma^+ = \bigoplus_{\lambda\in\Sigma_\sigma^+} \ker(A - \lambda\,\mathrm{Id}), \quad \mathbb{H}_{\sigma^\star}^+ = \bigoplus_{\lambda\in\Sigma_{\sigma^\star}^+} \ker(A - \lambda\,\mathrm{Id}).$$

The map P_σ^+ (resp. $P_{\sigma^\star}^+$) is then the orthogonal projection on $\mathbb{H}_\sigma^+$ (resp. $\mathbb{H}_{\sigma^\star}^+$) spanned by the eigenvectors associated to Σ_+^σ (resp. $\Sigma_+^{\sigma^\star}$).

Thus A_σ^+ (resp. $A_{\sigma^\star}^+$) can be viewed as the endomorphism on $\mathbb{H}_\sigma^+$ (resp. $\mathbb{H}_{\sigma^\star}^+$) associated to the diagonal matrix of eigenvalues Σ_σ^+ (resp. $\Sigma_{\sigma^\star}^+$).

We define the finite dimensional observer

$$\begin{aligned} &\widehat{z}_\star'(t) = A_{\sigma^\star}^+ \widehat{z}_\star(t) + \sum_{i=1}^{N_\sigma} \langle \widehat{z}_\star, \zeta_i\rangle_{\mathbb{H}} B_{\sigma^*}^+ w_i + \sum_{i=1}^{N_\sigma} \big\langle (C_{\sigma^\star}^+ \widehat{z}_\star - y), w_i^\star \big\rangle_{\mathbb{Y}} \chi_i,\\ &\widehat{z}_\star(0) = 0, \end{aligned} \tag{5.2}$$

where $(w_i^\star)_{1\leqslant i\leqslant N_\sigma} \subset \mathbb{Y}$, $(\chi_i)_{1\leqslant i\leqslant N_\sigma} \subset \mathbb{H}$, $(\zeta_i)_{1\leqslant i\leqslant N_\sigma} \subset \mathcal{D}(A^*)$ and $(w_i)_{1\leqslant i\leqslant N_\sigma} \subset \mathbb{U}$ are chosen as the previous section (see Eqs. (2.15–2.16) and Remark 2.3).

Using Theorem 1.2, we deduce the following result:

Theorem 5.1 *Let $\sigma > 0$ and $\sigma^\star > \sigma$. There exists a control v based on $\widehat{z}_\star$ solution of the system (5.2) where*

$$v(t) = \sum_{i=1}^{N_\sigma} \left(\int_\Omega \widehat{z}_\star(t)\zeta_i \, dx \right) w_i, \tag{5.3}$$

with $\zeta_i \in H^2(\Omega) \cap H_0^1(\Omega)$, $w_i \in L^2(\mathcal{O}')$, $i = 1, \ldots, N_\sigma$ such that for $\sigma^\star$ large enough, the coupled system (5.1) and (5.2) is exponentially stable that satisfies for $z^0 \in L^2(\Omega)$ the estimate

$$\|z(t)\|_{L^2(\Omega)} \leqslant Ce^{-\sigma t}\|z^0\|_{L^2(\Omega)}. \tag{5.4}$$

6 Conclusion

In this paper, we proposed a constructive method to stabilize a class of linear parabolic evolution systems $z' = Az + Bv$ by means of a finite dimensional control v, under the observation $y = Cz$. The control is based on the design of a Luenberger observer which can be infinite dimensional or finite dimensional of dimension large enough (under stronger assumptions). In both cases, we show that if (A, B) and (A, C) verify the Fattorini-Hautus Criterion, then we can construct an observer-based control v of finite dimension such that the closed-loop system is exponentially stable. The results are applied to study the stabilization of the diffusion system with Dirichlet boundary control and an internal observation. Possible future work is to implement this method numerically for the example studied here, or for more involved examples related to population dynamics, in the spirit of [16].

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